Int J Theor Phys (2010) 49: 395–401
DOI 10.1007/s10773-009-0215-7

Entanglement Evolution and Transfer in Non-Markovian Reservoirs

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Received: 11 September 2009 / Accepted: 1 December 2009 / Published online: 5 December 2009

Abstract We study the dynamics of two entangled two-level atoms A and B which locally interact with independent reservoirs a and b, respectively. We consider the non-Markovian effects of reservoirs on the entanglement revival for the remote parties AB, ab, Ab and Ba after a simultaneous entanglement sudden death. In addition, we investigate the dynamics of multipartite entanglement that is created along with the decay of pairwise entanglement.

Keywords Entanglement dynamics · Non-Markovian reservoir · Entanglement sudden death

1 Introduction

Entanglement is a unique feature of quantum mechanical systems with no classical analogue. In recent years entanglement has attracted extensive studies not only for fundamental reasons but also for practical purposes, as entanglement is realized an essential resource for various intriguing quantum computation (QC) and quantum information processing (QIC) [1]. However, in the process of entanglement distribution and qubits' manipulation, each qubit is unavoidably exposed to its own uncontrollable environment. This leads to local decoherences which will sooner or later spoil the necessary entanglement of the shared states. In the case where entangled quantum system interacts with external environment, the excitation emission is usually responsible for the entanglement decay of the system. Therefore, in one and the same environment, the system with different initial excitations would present different decay rate. In particular, the entanglement may decay to zero in finite time or infinite time. It was shown that a pair of entangled qubits with both of them can be simultaneously populated in excited states can become completely disentangled in a finite time under the influence of pure vacuum noise [2]. The phenomenon named as "entanglement sudden death" (ESD) [3–19] exhibit striking difference with the usual local decoherence in asymptotic time. An entangled state undergoing ESD would put a limitation to its usage time and is

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therefore less robust than the state without ESD. A natural question one usually asks on the ESD is related to where the lost entanglement finally go. To answer this question, in [20], the authors studied a two-qubit Bell-like entanglement interacting with independent reservoirs and showed that the corresponding reservoirs can experience entanglement sudden birth (ESB) whenever the ESD occur for the qubits system. That is, the initial entanglement of the system of interest can be transferred into the corresponding environments. However, the entanglement of interested system and the corresponding environments can simultaneously be zero sometimes. Therefore, one should consider other entanglement forms created newly in time evolution.

In [9], the authors studied a closed system in which two initially entangled two-level atoms A and B are independently coupled with separate cavity fields a and b, respectively, but there are no interactions at all between the subsystems Aa and Bb. They considered two types of Bell-like states $|\phi_{AB}\rangle \sim |ee\rangle + |gg\rangle$ and $|\psi_{AB}\rangle \sim |eg\rangle + |ge\rangle$ as atomic initial states and found out that $|\phi_{AB}\rangle$ undergoes ESD but $|\psi_{AB}\rangle$ dose not in the time-evolving process [9]. Their results revealed that under one and the same kind of environment, the disentanglement dynamics might be dependent on the type of initial entangled state. In this paper we shall study the similar model but in the non-Markovian reservoirs. We shall pay our attentions on the entanglement transfer from initial pairwise entanglement to genuine multipartite entanglement created in time evolution. We show that there exist a time window in which all the pairwise entanglement of remote parties, i.e., AB, ab, Ab and Ba, can vanish simultaneously for certain parameters. In order to explain this phenomenon, we resort to the multipartite entanglement that might be created during the time evolution and correlate now the two independent locations in place of the pairwise entanglement.

2 Physical Model and Entanglement Measures

Consider a quantum system consists of two two-level atoms A and B locally interacting with two independent zero temperature reservoirs a and b, respectively. The two parts Aa and Bb have no any interactions except the initial correlations originating from atoms' entanglement, thus the evolution of the whole system can be obtained from that of the independent atom-reservoir interaction which can be described as ($\hbar = 1$)

$$\hat{H} = \omega\hat{\sigma}_+\hat{\sigma}_- + \sum_{i=1}^{N}\omega_i\hat{a}_i^+\hat{a}_i^- + \sum_{i=1}^{N}g_i(\hat{\sigma}_-\hat{a}_i^+ + \hat{\sigma}_+\hat{a}_i^-), \tag{1}$$

where $\hat{a}_i^+$ and $\hat{a}_i^-$ are the creation and annihilation operators of the mode i of the reservoir; $\hat{\sigma}_+ = |1\rangle\langle 0|$, $\hat{\sigma}_- = |0\rangle\langle 1|$ and ω are the inversion operators and transition frequency of the atom; ω_i and g_i are the frequency of the mode i of the reservoir and its coupling strength with the atom.

Governed by the above Hamiltonian (1), a single atom-reservoir in the initial state $|1\rangle \otimes |\bar{\mathbf{0}}_r\rangle$ with $|\bar{\mathbf{0}}_r\rangle = \prod_{i=1}^{N}|0_i\rangle_r$ will evolve into

$$|\phi(t)\rangle = C_0(t)\,|1\rangle\left|\bar{\mathbf{0}}\right\rangle_r + \sum_{i=1}^{N}C_i(t)\,|0\rangle\,|1_i\rangle_r, \tag{2}$$

where $|1_i\rangle_r$ is the state of the reservoir with only one exciton in the ith mode. Setting the detuning between atom and the ith mode of reservoir $\delta_i = \omega - \omega_i$, the equations for the probability amplitudes take the form

$$\dot{C}_0(t) = -i \sum_{i=1}^{N} g_i e^{i\delta_i t} C_i(t), \tag{3}$$

$$\dot{C}_i(t) = -i g_i^* e^{-i\delta_i t} C_0(t). \tag{4}$$

Integrating (4) and inserting its solution into (3), one gets a integro-differential equation for the amplitudes $C_0(t)$,

$$\dot{C}_0(t) = -\sum_{i=1}^{N} |g_i|^2 \int_0^t dt_1 C_0(t_1). \tag{5}$$

In the continuum limit for the reservoir spectrum the sum over the modes is replaced by the integral $\sum_{i=1}^{N} |g_i|^2 \to \int d\omega J(\omega)$, where $J(\omega)$ is the reservoir spectral density. In the following we first consider the structured reservoir as the electromagnetic field inside a lossy cavity. In this case, the fundamental mode supported by the cavity displays a Lorentzian broadening due to the non-perfect reflectivity of the cavity mirrors. Then the spectrum of the field inside the cavity can be modeled as

$$J(\omega) = \frac{R^2}{\pi} \frac{\Gamma}{(\omega - \omega_c)^2 + (\Gamma)^2}, \tag{6}$$

where R is related to the atom-reservoir coupling constant, Γ defines the spectral width of the coupling and is connected to the reservoir correlation time τ_B by the relation $\tau_B \approx 1/\Gamma$, and ω_c is the fundamental frequency of the cavity. A weak and a strong coupling regime can be distinguished for the single atom dynamics. The weak regime corresponds to the case $\Gamma/R > 2$ in which the relaxation time of atom is greater than the reservoir correlation time and the dynamics is essentially Markovian. The strong-coupling regime corresponds to $\Gamma/R < 2$, in which the reservoir correlation time is greater or of the same order of the relaxation time and non-Markovian effects become relevant.

Through introducing the correlation function $f(t - t_1) = \int d\omega J(\omega) e^{i(\omega_c - \omega)(t - t_1)}$ and performing Laplace transform of (5), one can obtain a formal solution for the amplitude

$$C_0(t) = e^{-(\Gamma - i\delta)t/2} \left[\cosh\left(\frac{\Omega t}{2}\right) + \frac{\Gamma - i\delta}{\Omega} \sinh\left(\frac{\Omega t}{2}\right) \right], \tag{7}$$

where $\Omega = \sqrt{(\Gamma - i\delta)^2 - 4R^2}$. In terms of the reservoir's normalized collective state with one excitation $|\overline{\mathbf{1}}\rangle_r = \frac{1}{C(t)} \sum_{i=1}^{N} C_i(t) |1_i\rangle_r$, (2) can be rewritten as

$$|\phi_t\rangle = C_0(t)\, |1\rangle \left|\overline{\mathbf{0}}\right\rangle_r + C(t)\, |0\rangle \left|\overline{\mathbf{1}}\right\rangle_r, \tag{8}$$

where $C(t) = \sqrt{1 - |C_0(t)|^2}$. Note that each reservoir can be either in the vacuum or in a one-excitation state so it can also be regarded as a qubit.

In two-qubit domains, there exist a number of good measures of entanglement such as concurrence [21], negativity [22–25], entropy of formation, Schmitdt number, etc. Although the aforementioned entanglement measures may be somewhat different quantitatively, they are equivalent qualitatively to each other in the sense that all of them are equal to zero for unentangled (i.e., separable with respect to the bipartite partition) states. Here we adopt negativity $\mathcal{N}$ [22] as the measure of entanglement, which is defined as two times the absolute value of the sum of the negative eigenvalues of the partially transposed matrix ρ^{T_A} of ρ, where $\langle i_A j_B|\rho^{T_A}|k_A l_B\rangle = \langle k_A j_B|\rho|i_A l_B\rangle$. For short:

$$\mathcal{N}(\rho) = 2\max\{0, -\lambda_{\min}\}, \tag{9}$$

where $\lambda_{\min}$ is the lowest eigenvalue of ρ^{T_A}.

For certain multiqubit pure entangled state, there are also potential measures, such as the N-tangle proposed in [26]. The N-tangle is equal to the square of the concurrence for a system of two qubits, and is equal to the "residual entanglement" for systems of three qubits. The N-tangle is also equal to a generalization of the concurrence squared for even N. The N-tangle is defined as [26],

$$\begin{aligned}\tau_{1\ldots N} = 2|&\Sigma a_{\alpha_1\ldots\alpha_N} a_{\beta_1\ldots\beta_N} a_{\gamma_1\ldots\gamma_N} a_{\delta_1\ldots\delta_N}\\ &\times \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\cdots\epsilon_{\alpha_{N-1}\beta_{N-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\\ &\times \epsilon_{\gamma_{N-1}\delta_{N-1}}\epsilon_{\alpha_N\gamma_N}\epsilon_{\beta_N\delta_N}|,\end{aligned} \tag{10}$$

where the a terms are the coefficients in the standard basis defined by $|\psi\rangle = \Sigma_{i_1\ldots i_N} a_{i_1\ldots i_N}|i_1 i_2\ldots i_N\rangle$, and $\epsilon_{01} = -\epsilon_{10} = 1$ and $\epsilon_{00} = -\epsilon_{11} = 0$.

3 Entanglement Dynamics

In what follows, we consider qubits A and B are prepared initially in the form

$$|\psi(0)\rangle_{AB} = \cos\alpha|11\rangle_{AB} + \sin\alpha|00\rangle_{AB}, \tag{11}$$

while both reservoirs a and b are in the vacuum states $|\overline{\mathbf{00}}\rangle_{ab}$. After time $t > 0$, the total system will evolve into

$$\begin{aligned}|\psi(t)\rangle_{AaBb} = {}&\cos\alpha C_0(t)^2|1\overline{\mathbf{0}}1\overline{\mathbf{0}}\rangle + \cos\alpha C_0(t)C(t)|0\overline{\mathbf{1}}1\overline{\mathbf{0}}\rangle + \cos\alpha C_0(t)C(t)|1\overline{\mathbf{0}}0\overline{\mathbf{1}}\rangle\\ &+ \cos\alpha C(t)^2|0\overline{\mathbf{1}}0\overline{\mathbf{1}}\rangle + \sin\alpha|0\overline{\mathbf{0}}0\overline{\mathbf{0}}\rangle.\end{aligned} \tag{12}$$

To study the entanglement evolution of any bipartite subsystems we should first know its reduced density matrix which can be obtained by tracing the total state (12) over the degrees of freedom of the remaining parties. By virtue of (9), we can obtain the pairwise entanglement of each qubit-pair in remote locations, i.e., $\mathcal{N}_{AB}$, $\mathcal{N}_{ab}$ and $\mathcal{N}_{Ab} = \mathcal{N}_{Bb}$. In Fig. 1 we plot the entanglement evolution of atoms AB, reservoirs ab and remote atom-reservoir $Ab(Ba)$ with $\alpha = 3/10$ in non-Markovian regime. From Fig. 1 one can see that atoms AB can exhibit finite-time complete disentanglement. At the same time, the entanglement of atoms AB can revive after a finite interval due to the non-Markovian effects of the reservoirs. Because of entanglement transfer, independent reservoirs ab and remote atom-reservoir can be entangled in the time evolution. Remarkably, we find that all the pairwise entanglement in the remote locations can vanish simultaneously and the bipartite correlations between them seem to disappear during the period from t_1 to t_2, as shown in Fig. 1. However, we know that the interactions between atoms and reservoirs are local which would not lead to the loss of correlation between independent locations Aa and Bb. Hence, the initial entanglement of atoms should be transferred into other forms which account for the current correlations of Aa and Bb. Therefore, we should consider the creation of genuine four-partite entanglement

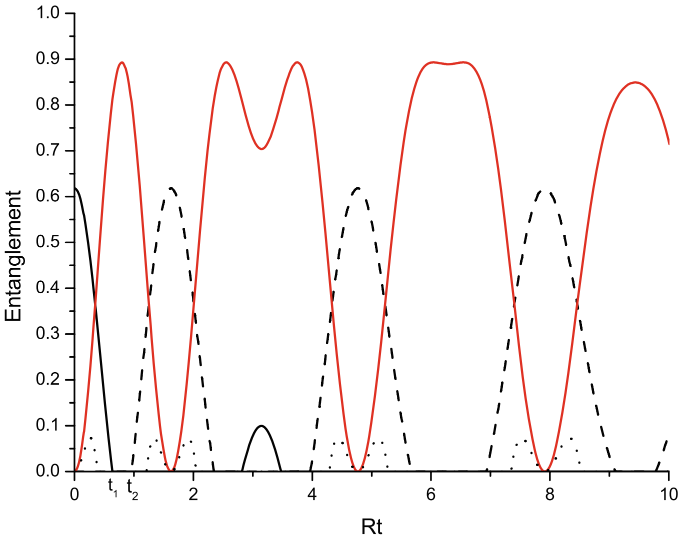

Fig. 1 (Color online) The pairwise entanglement of AB (*black solid line*), ab (*black dashed line*), $Ab(Ba)$ (*black dotted line*) and genuine four-partite entanglement C_{AaBb} (*red solid line*) as a function of rescaled time Rt for $\alpha = 3/10$, $\Gamma = 0.1R$ and $\delta = 0$

and its dynamical behaviors. By virtue of the definition of N-tangle (10), we can obtain the four-partite entanglement of the total system $AaBb$ as

$$C_{AaBb} = 4\cos^2\alpha |C_0(t)|^2 |C(t)|^2. \tag{13}$$

In Fig. 1, we also plot C_{AaBb} as a function of rescaled time Rt. It is worth noting that the genuine four-partite entanglement not only exists in the simultaneous ESD period but also can take its maximal value. Therefore it can be said that the genuine four-partite entanglement is created along with the decay of the remote pairwise entanglement. However, the genuine four-partite entanglement will decay and eventually vanish. The initial entanglement of atoms shall be transferred into the reservoirs in the long time limit. This can shown in Fig. 2 for the entanglement evolution of C_{AaBb} and reservoirs ab.

4 Conclusion

In conclusion, we have investigated entanglement dynamics of a four-qubit system in non-Markovian reservoirs where two atoms A and B are locally coupled with independent reservoirs a and b, respectively. Atoms AB are initially prepared in an entangled state while the independent reservoirs ab in vacuum states. We find that the pairwise entanglement of remote parties, i.e. AB, ab, Ab and Ba can experience simultaneous ESD and keep zero for a finite time before being recovered by memory effects of non-Marovians. The simultaneous ESD for the remote pairwise entanglement seem to imply the vanish of correlation between independent locations Aa and Bb, but this is not the case since the interactions between atom and cavity are local and can not change the initial correlation. In order to explain this phenomenon, we resort to consider other newly generated entanglement forms which can account for the correlations between Aa and Bb. We have considered the genuine four-partite entanglement created in time-evolution. As a result, we find that in the simultaneous

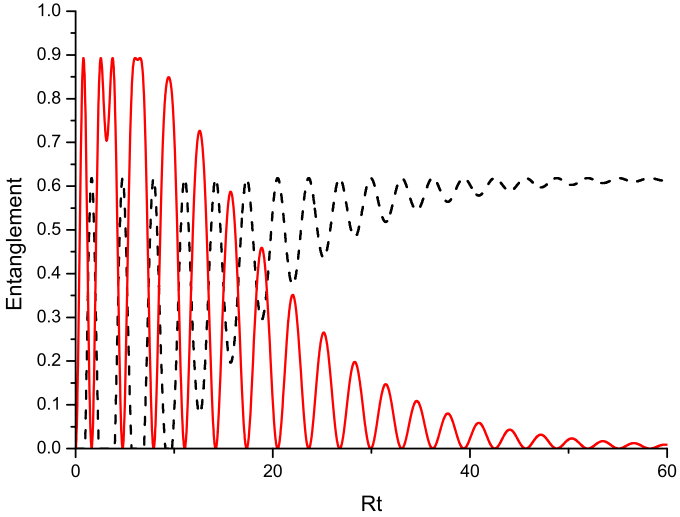

Fig. 2 (Color online) The genuine four-partite entanglement C_{AaBb} (*red solid line*) versus entanglement of two reservoirs *ab* (*black dashed line*) as a function of rescaled time Rt for $\alpha = 3/10$, $\Gamma = 0.1R$ and $\delta = 0$

ESD period, the genuine four-partite entanglement is not zero, meaning that the correlation between remote locations Aa and Bb still exists. Our study illustrates that during time evolution the initial pairwise entanglement can be transferred into other multipartite forms. Multipartite entanglement and its dynamical properties are an open question in quantum information and computation, our study provide an meaningful insight to this issue from an entanglement transfer perspective.

Acknowledgement This work is supported by National Natural Science Foundation of China under Grant No. 10774088.

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